CASCADED ON-OFF SOURCES

Here I define a Markovian cascaded on-off source with parameters (m, λ, μ). The parameter m gives the long run mean rate (in bits/sec), the on times are determined by the parameter λ, the off times are determined by the parameters λ and μ. Many variations of the following construction are possible. I describe the simplest.

Let $\{X_n(t), t \geq 0\}$ $(n = 1, 2, 3, ...)$ be a sequence of independent stationary Continuous Time Markov Chains (CTMC) on state space $\{0, 1\}$ with rate matrix

$$\begin{bmatrix} -2^{n-1}\mu & 2^{n-1}\mu \\ 2^{n-1}\lambda & -2^{n-1}\lambda \end{bmatrix}.$$

Define

$$Y_n(t) = \prod_{i=1}^{n} X_i(t), \quad t \geq 0, \;\; n = 1, 2, 3, ...$$

Next define

$$Z_n(t) = m \left(\frac{\lambda + \mu}{\mu} \right)^n Y_n(t), \quad t \geq 0, \;\; n = 1, 2, 3, ...$$

$\{Z_n(t), t \geq 0\}$ is a (stationary) instantaneous rate process of a level-n cascaded on-off source. Notice that for any given $n \geq 1$, $\{Z_n(t), t \geq 0\}$ is an on-off source with iid exponential on times with parameter (i.e., 1/mean) $(2^n - 1)\lambda$, and iid (non-exponential) off times with mean $[(\frac{\lambda+\mu}{\mu})^n - 1]/((2^n - 1)\lambda)$, and long-run mean rate m.

An interesting observation: the mean on time goes to 0 as n goes to infinity. As $n \to \infty$, the mean off time goes to zero if $\lambda < \mu$, goes to infinity if $\lambda > \mu$ and stays constant $(= 1/\lambda)$ if $\lambda = \mu$. However, the ratio of mean off times to the mean on times always goes to infinity. Hence I suspect the limiting behavior of the Z_n process (as $n \to \infty$) may depend on the relative magnitudes of λ and μ, displaying fractal behavior when $\lambda < \mu$.

The On times: To describe the on times consider the n-dimensional CTMC $\{(X_1(t), X_2(t), ..., X_n(t)), t \geq 0\}$. Let a be a n-vector of all ones. The Z_n process is in state 1 if and only if the multidimensional process is in state a. The rate at which it exits state a is $\lambda + 2\lambda + 4\lambda + ... + 2^{n-1}\lambda = (2^n - 1)\lambda$. Hence the on times are iid exponential rvs with mean $1/((2^n - 1)\lambda)$.

The Off times: Let a_i be a n-vector whose ith coordinate is zero and all other coordinates are 1 $(i = 1, 2, ..., n)$. When the multidimentional process leaves state a (the Z_n process enters state 0) it enters state a_i with rate $2^{i-1}/(2^n -$

1). Let T_i be the first passage time of the multidimensional CTMC from state a_i into state a. Then the off time is T_i with probability $2^{i-1}/(2^n-1)$.

The LST of T. Let $\rho = \lambda/\mu$, and define

$$h(t) = P(X_j(t) = 1, j = 1, 2, ..., n | X_j(0) = 1, j = 1, 2, ..., n)$$
$$= \left(\frac{\mu}{\lambda+\mu}\right)^n \prod_{j=1}^{n} \left(1 + \rho e^{-2^{j-1}(\lambda+\mu)t}\right),$$

and

$$h_i(t) = P(X_j(t) = 1, j = 1, 2, ..., n | X_j(0) = 1, j \neq i, X_i(t) = 0)$$
$$= \left(\frac{\mu}{\lambda+\mu}\right)^n \left(1 - e^{-2^{i-1}(\lambda+\mu)t}\right) \prod_{j=1, j\neq i}^{n} \left(1 + \rho e^{-2^{j-1}(\lambda+\mu)t}\right).$$

Let $\tilde{h}(s)$ be the Laplace Transform of $h(t)$ and $\tilde{h}_i(s)$ be the Laplace Transform of $h_i(t)$. Then

$$\phi_i(s) = E(e^{-sT_i}) = \frac{\tilde{h}_i(s)}{\tilde{h}(s)}.$$

$h_i(t)$ also provides a simple upper bound on the distribution of T_i. The probability distribution and the LSt of of the off time T is given by

$$P(T \leq t) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^n - 1} P(T_i \leq t),$$

$$\phi(s) = E(e^{-sT}) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^n - 1} \phi_i(s).$$

The pdf of T. Let Q be the $2^n \times 2^n$ rate matrix of the multidimensional CTMC. Let $\hat{Q}$ be a modified matrix obtained from Q by deleting the row and column corresponding to the state a. Also, let α be a row vector of size $2^n - 1$, representing the initial state of the multidimensional CTMC. Set the probability that the multidimensional CTMC starts in state a_i $(i = 1, 2, ..., n)$ to be $2^{i-1}/(2^n-1)$. Let e be a column vector of length $2^n - 1$ with all components equal to 1. The cdf of T is given by:

$$P(T \leq t) = 1 - \alpha \exp(\hat{Q}t)e.$$

Since the off time starts whenever the multidimensional CTMC jumps out of state a, we see that the successive off times are iid.

The autocovariance function: can be easily calculated to be

$$\mathrm{Cov}(Z_n(t), Z_n(0)) = m^2 \left[\prod_{i=1}^{n} \left(1 + \rho e^{-2^{i-1}(\lambda+\mu)t}\right) - 1\right].$$

Thus the autocovariance dies off exponentially, dominated by the exponential decay rate of $\lambda+\mu$. Thus, clearly the process will not display long range dependence! To get that (I feel that) we need to start with stationary semi-Markov processes with off time distributions having power laws.

Queueing Analysis. Since the level-n cascaded process is an on-off process with iid exponential on times and generally distributed off times, one can use the existing theory to do the queueing analysis. Let c be the output rate of the buffer, and

$$r_n = m\left(\frac{\lambda+\mu}{\mu}\right)^n,$$

$$\lambda_n = (2^n - 1)\lambda.$$

Let η be the smallest solution to

$$\phi(\eta c) = 1 - \frac{\eta(r_n - c)}{\lambda_n}.$$

Let B be the steady state buffer content. We have

$$P(B > x) \leq e^{-\eta x}.$$

Parameter Estimation: The trace data gives the following.

1. The peak rate r_{peak} is the true transmission rate of the medium, in bits per second.
2. N = total number of packets in the trace
3. T_i = start time of the ith packet, in secs. $(i = 1, 2, ..., N)$
4. S_i = size of the ith packet, in bits. $(i = 1, 2, ..., N)$

The estimation procedure is:

1. Estimate the mean rate r_{mean} by
$$\hat{r}_{mean} = \frac{\sum_i S_i}{T_N}.$$
2. Estimate the mean on time τ_{on} by
$$\hat{\tau}_{on} = \frac{\sum_i S_i}{N r_{peak}}.$$
3. Estimate the mean off time τ_{off} by
$$\hat{\tau}_{off} = \frac{T_N}{N} - \hat{\tau}_{on}.$$

4. For each $n = 1, 2, 3...$ compute

$$\hat{\lambda}_n = \frac{1}{\hat{\tau}_{on}(2^n - 1)},$$

$$\hat{\mu}_n = \frac{\hat{\lambda}_n}{\left[\frac{\hat{\tau}_{off}}{\hat{\tau}_{on}} + 1\right]^{1/n} - 1}.$$

5. There is no unique way to choose n. Try various values, and generate simultaed samples. Pick the smallest n that produces "reasonable" looking sample paths.

What can we do with this?

1. Plot the sample paths to see if they "look" reasonable.

2. Estimate m, λ and μ.

3. Study the fractal properties of the limit of Z_n as $n \to \infty$.