ORIE 779: Functional Data Analysis

From last meeting

Finished Robust FDA: Elliptical Mean & PCA

- Cornea Data
- Parabolas with 2 outliers

From last meeting (cont.)

Started detailed look at PCA

Three important (and interesting) viewpoints:

1. Mathematics

2. Numerics

3. Statistics

Linear Algebra Review, (cont.)

Norm of a vector:

- in $\Re^d$, $\quad \|\underset{\rightarrow}{x}\| = \left(\sum_{j=1}^{d} x_j^2 \right)^{1/2} = \left(\underset{\rightarrow}{x}^t \underset{\rightarrow}{x} \right)^{1/2}$

- Idea: "length" of the vector

- Note: $\exists$ strange properties for high d,
  e.g. "length of diagonal of unit cube" = $\sqrt{d}$

- "length normalized vector": $\dfrac{\underset{\rightarrow}{x}}{\|\underset{\rightarrow}{x}\|}$
  (has length one, this is on surface of unit sphere)

- get "distance" as: $d\left(\underset{\rightarrow}{x}, \underset{\rightarrow}{y}\right) = \|\underset{\rightarrow}{x} - \underset{\rightarrow}{y}\| = \sqrt{\left(\underset{\rightarrow}{x} - \underset{\rightarrow}{y}\right)^t \left(\underset{\rightarrow}{x} - \underset{\rightarrow}{y}\right)}$

Linear Algebra Review, (cont.)

Inner (dot, scalar) product:

- for vectors $\underset{\rightarrow}{x}$ and $\underset{\rightarrow}{y}$, $\langle \underset{\rightarrow}{x}, \underset{\rightarrow}{y} \rangle = \sum_{j=1}^{d} x_j y_j = \underset{\rightarrow}{x}^t \underset{\rightarrow}{y}$

- related to norm, via $\|\underset{\rightarrow}{x}\| = \sqrt{\langle \underset{\rightarrow}{x}, \underset{\rightarrow}{x} \rangle} = \sqrt{\underset{\rightarrow}{x}^t \underset{\rightarrow}{x}}$

- measures "angle between $\underset{\rightarrow}{x}$ and $\underset{\rightarrow}{y}$" as:

$$angle(\underset{\rightarrow}{x}, \underset{\rightarrow}{y}) = \cos^{-1}\left(\frac{\langle \underset{\rightarrow}{x}, \underset{\rightarrow}{y} \rangle}{\|\underset{\rightarrow}{x}\| \cdot \|\underset{\rightarrow}{y}\|} \right) = \cos^{-1}\left(\frac{\underset{\rightarrow}{x}^t \underset{\rightarrow}{y}}{\sqrt{\underset{\rightarrow}{x}^t \underset{\rightarrow}{x} \cdot \underset{\rightarrow}{y}^t \underset{\rightarrow}{y}}} \right)$$

- key to "orthogonality", i.e. "perpendicularity":

$\underset{\rightarrow}{x} \perp \underset{\rightarrow}{y}$ if and only if $\langle \underset{\rightarrow}{x}, \underset{\rightarrow}{y} \rangle = 0$

Linear Algebra Review, (cont.)

Orthonormal basis $\underset{\rightarrow}{v_1},\ldots,\underset{\rightarrow}{v_n}$:

- All ortho to each other, i.e. $\left\langle \underset{\rightarrow}{v_i}, \underset{\rightarrow}{v_{i'}} \right\rangle = 0,$ for $i \neq i'$

- All have length 1, i.e. $\left\langle \underset{\rightarrow}{v_i}, \underset{\rightarrow}{v_i} \right\rangle = 1,$ for $i = 1,\ldots,n$

- "Spectral Representation": $\underset{\rightarrow}{x} = \sum_{i=1}^{n} a_i \underset{\rightarrow}{v_i}$ where $a_i = \left\langle \underset{\rightarrow}{x}, \underset{\rightarrow}{v_i} \right\rangle$

  check: $\left\langle \underset{\rightarrow}{x}, \underset{\rightarrow}{v_i} \right\rangle = \left\langle \sum_{i'=1}^{n} a_{i'} \underset{\rightarrow}{v_{i'}}, \underset{\rightarrow}{v_i} \right\rangle = \sum_{i'=1}^{n} a_{i'} \left\langle \underset{\rightarrow}{v_{i'}}, \underset{\rightarrow}{v_i} \right\rangle = a_i$

- Matrix notation: $\underset{\rightarrow}{x} = B\underset{\rightarrow}{a}$ where $\underset{\rightarrow}{a}^t = \underset{\rightarrow}{x}^t B$ i.e. $\underset{\rightarrow}{a} = B^t \underset{\rightarrow}{x}$

- $\underset{\rightarrow}{a}$ is called "transform (e.g. Fourier, wavelet) of $\underset{\rightarrow}{x}$"

Linear Algebra Review, (cont.)

Parseval identity, for $\underrightarrow{x}$ in subsp. gen'd by o. n. basis $\underrightarrow{v_1}, \ldots, \underrightarrow{v_n}$:

- $\left\|\underrightarrow{x}\right\|^2 = \sum_{i=1}^{n} \left\langle \underrightarrow{x}, \underrightarrow{v_i} \right\rangle^2 = \sum_{i=1}^{n} a_i^2 = \left\|\underrightarrow{a}\right\|^2$

- Pythagorean theorem

- "Decomposition of Energy"

- ANOVA - sums of squares

- Transform, $\underrightarrow{a}$, has same length as $\underrightarrow{x}$, i.e. "rotation in $\Re^d$"

Linear Algebra Review, (cont.)

Projection of a vector $\underset{\rightarrow}{x}$ onto a subspace V:

- Idea: member of V that is closest to $\underset{\rightarrow}{x}$ (i.e. "approx'n")

- Find $P_V \underset{\rightarrow}{x} \in V$ that solves: $\min_{v \in V} \left\| \underset{\rightarrow}{x} - \underset{\rightarrow}{v} \right\|$ ("least squares")

- For inner product (Hilbert) space: exists and is unique

- General solution in $\Re^d$: for basis matrix B_V

$$P_V \underset{\rightarrow}{x} = B_V \left(B_V^t B_V \right)^{-1} B_V^t \underset{\rightarrow}{x}$$

- So "proj'n operator" is "matrix mult'n": $P_V = B_V \left(B_V^t B_V \right)^{-1} B_V^t$

  (thus projection is another linear operation)

  (note same operation underlies "least squares")

[go to next part]