Linear Algebra Review, (cont.)

Projection using orthonormal basis $\underrightarrow{v_1}, \ldots, \underrightarrow{v_n}$:

- Basis matrix is "orthonormal": $B_V^t B_V = I_{n \times n}$

$$\begin{pmatrix} \underrightarrow{v_1}^t \\ \vdots \\ \underrightarrow{v_n}^t \end{pmatrix} \begin{pmatrix} \underrightarrow{v_1} & \cdots & \underrightarrow{v_n} \end{pmatrix} = \begin{pmatrix} \langle \underrightarrow{v_1}, \underrightarrow{v_1} \rangle & \cdots & \langle \underrightarrow{v_1}, \underrightarrow{v_n} \rangle \\ \vdots & \ddots & \vdots \\ \langle \underrightarrow{v_n}, \underrightarrow{v_1} \rangle & \cdots & \langle \underrightarrow{v_n}, \underrightarrow{v_n} \rangle \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

- So $P_V \underrightarrow{x} = B_V \left(B_V^t \underrightarrow{x} \right)$ = Recon(Coeffs of $\underrightarrow{x}$ "in V dir'n")

- For "orthogonal complement", $V^{\perp}$,

$$\underrightarrow{x} = P_V \underrightarrow{x} + P_{V^{\perp}} \underrightarrow{x} \quad \text{and} \quad \left\| \underrightarrow{x} \right\|^2 = \left\| P_V \underrightarrow{x} \right\|^2 + \left\| P_{V^{\perp}} \underrightarrow{x} \right\|^2$$

- Parseval inequality: $\left\| P_V \underrightarrow{x} \right\|^2 \leq \left\| \underrightarrow{x} \right\|^2 = \sum_{i=1}^{n} \left\langle \underrightarrow{x}, \underrightarrow{v_i} \right\rangle^2 = \sum_{i=1}^{n} a_i^2 = \left\| \underrightarrow{a} \right\|^2$

Linear Algebra Review, (cont.)

(Real) Unitary Matrices: $U_{d \times d}$ with $U^t U = I$

- Orthonormal basis matrix (so all of above applies)
- Follows that $UU^t = I$
- (since have full rank, so U^{-1} exists …)
- Linear transform (mult'n by U) is like "rotation" of $\Re^d$
- But also includes "mirror images"

Linear Algebra Review, (cont.)

Singular Value Decomposition:

For a matrix $X_{d \times n}$

Find a diagonal matrix $S_{d \times n}$,

with entries $s_1, \ldots, s_{\min(d,n)}$ - called singular values

And unitary (rotation) matrices $U_{d \times d}$, $V_{n \times n}$ (recall $U^t U = V^t V = I$)

so that $X = USV^t$

Linear Algebra Review, (cont.)

Intuition behind Singular Value Decomposition:

For X a "linear transformation" (via matrix multiplication)

- $X \cdot \underset{\rightarrow}{v} = \left(U \cdot S \cdot V^t\right) \cdot \underset{\rightarrow}{v} = U \cdot \left(S \cdot \left(V^t \cdot \underset{\rightarrow}{v}\right)\right)$
- First "rotate"
- Second "rescale coordinate axes (by s_i)"
- Third "rotate again"
- i.e. have "diagonalized the transformation"

[go to next part]