Statistics 322, Statistical Computing, Home Work 1

Handed Out: August 22, 1993 Due: August 29, 1993

1. Comparison of Sequences:

Given constants $\alpha, \beta > 0$, consider the "power sequence":

$$\{a_j\}_{j=2}^{\infty} = \{j^{-\alpha}\}_{j=2}^{\infty} = \{2^{-\alpha}, 3^{-\alpha}, 4^{-\alpha} \ldots\},$$

and the "exponential sequence":

$$\{b_j\}_{j=2}^{\infty} = \{e^{-\beta j}\}_{j=2}^{\infty} = \{e^{-2\beta}, e^{-3\beta}, e^{-4\beta} \ldots\}.$$

a. Both of these sequences tend to 0 in the limit. They both tend to 0 more slowly when α or β is closer to 0. An interesting bit of "mathematical folklore" is that, regardless of the values of α and β (> 0), the exponential sequence tends to 0 faster. Verify this by calculating an appropriate limit.

(Hint: why is it not enough to look at the difference?)

b. The above limiting result reflects what happens eventually, however, for α large (i.e. the power sequence tends to 0 rapidly), and β small, (i.e. the exponential sequence tends to zero slowly), the power sequence will be smaller for quite a while, as $j \to \infty$. Write a MATLAB script, which shows you the values of the two sequences, in the case $\alpha = 1$, $\beta = 0.1$. Use it to find the first j where $a_j > b_j$. Do not turn in an output with a big sequence, but just do some trial and error yourself. Either write by hand, or else printout the answer in a simple way that is clear (note this only needs the values of $a_j, b_j, a_{j+1}, b_{j+1}$ for a carefully chosen choice of j). Do not turn in the program listing either.

(Suggestion: have α and β as variables in the program, that are defined on separate lines, because this makes the program easier to change for upcoming work. Also for later parts of this assignment, it will be convenient to have the starting point, jump, and number of terms in the sequence as variables, that you can twiddle, depending on what part of the sequence you want to look at. Experiment with the MATLAB commands `>> 1:5` and `>> 2:4:14` to see how to generate sequence vectors. Also useful is "component wise power": `x.^y`)

IMPORTANT NOTE: At this point you are NOT supposed to have your program find the correct parameter values (e.g. with IF statements, or whatever). You should do this ONLY by repeated trial and error modification of your progam with various parameter values (we will cover the more sophisticated stuff soon).

(Suggestion: think carefully about what you want to see appearing on the screen. For example, for each j, you may be interested in seeing a row on the screen which shows you $j, a_j, b_j, a_j - b_j$ and a_j / b_j.)

c. Now change the parameters to $\alpha = 10$, $\beta = .0001$. Again find the first j where $a_j > b_j$, and do not turn in long output, just the answer.

> (Comment: you will probably not be able to do this with a sequence starting at 2, with jump 1, because the size of the biggest matrix you can study is limited by the amount of memory on your machine. Hence you should fiddle a bit with sequence parameters, to only look at parts of the sequence at a time. Fiddling with both the starting point and the jump is very useful.)

d. Do the same as above, for the values $\alpha = 1000$, $\beta = .0001$.

> Comment: you will need extra care here. The reason is that even for $j = 10$ (but clearly MUCH bigger values need to be considered), $a_j = 10^{-1000}$ is a number that is far too close to 0 for the computer to handle (around $10^{\pm 300}$ is the limit for MATLAB, and of course that is not 300 significant digits!). This is usually not a serious practical limitation, because the number of atoms in the universe is thought to be along the lines of 10^{70}. However, numbers like 10^{-1000} can still be handled by the numerical device of "putting things on the correct scale", which in this case (and many others) is done by working with logarithms. Compare $\ln(a_j)$ with $\ln(b_j)$ to answer this part of the problem.

e. You may have noticed in the above that the ratio a_j/b_j seems to start decreasing, then at some point begins to increase, and then seems to increase forever after that. Verify analytically that this is true, and find the "change point" in terms of α and β.

2. Averaging Sequences:

Now consider finite versions of the "power sequence", in the special case $\alpha \in (0, 1)$:

$$\{a_j\}_{j=2}^{n} = \{j^{-\alpha}\}_{j=2}^{n} = \{2^{-\alpha}, 3^{-\alpha}, \dots, n^{-\alpha}\}$$

and think about "averaging" these $n-1$ numbers. There are many ways to do this, including: the "arithmetic mean":

$$\bar{a}_{n,A} = \tfrac{1}{n-1} \Sigma_{j=2}^{n} a_j ,$$

the "geometric mean":

$$\bar{a}_{n,G} = \left(\Pi_{j=2}^{n} a_j \right)^{\frac{1}{n-1}} = e^{\frac{1}{n-1} \Sigma_{j=2}^{n} \ln(a_j)} ,$$

and the "harmonic mean":

$$\bar{a}_{n,H} = \left(\tfrac{1}{n-1} \Sigma_{j=2}^{n} (1/a_j) \right)^{-1} .$$

a. A feature of the power sequence is that "most of the entries are not far from the last entry", in the sense that each of these averages has the asymptotic representation:

$$a_{n,\cdot} \sim C_{\alpha,\cdot} n^{-\alpha},$$

where $\sim$ means that the ratio tends to 1. In the case of the arithmetic mean, this can be proven through the bounds:

$$\frac{1}{n-1}\left(\frac{n^{1-\alpha}-1}{1-\alpha} + n^{-\alpha} - 1\right) \le \bar{a}_{n,A} \le \frac{n^{1-\alpha}-1}{(n-1)(1-\alpha)} \quad .$$

Prove this and give the value of $C_{\alpha,A}$.

(hint: do a graphical comparison of the sequence to an appropriate "continuous version", in the same way as done for the "integral test" for convergence of series.)

b. Write a GAUSS program which constructs a table that shows, in the special cases $n = 10,\ 100$ and 1000, and $\alpha = .01,\ .1,\ .5$ and $.9$, how the values of $\bar{a}_{n,A}$, $C_{\alpha,A}\, n^{-\alpha}$ and the upper and lower bounds in part a compare. Turn in this table, together with discussion of why some cases are much better than others.

(suggestions: Make your program write out to a file, which can then be printed, as done in 322eg1.m. For taking averages, experiment a bit with different matrix arguments in the MATLAB command >> `mean()`.

c. Find, but do not submit a detailed proof, the values of $C_{\alpha,G}$ and $C_{\alpha,H}$. Show that for all $\alpha \in (0, 1)$, $C_{\alpha,H} \le C_{\alpha,G} \le C_{\alpha,A}$.

d. The phenomenon in the inequality of part c is actually a special case of a very general relationship. Show that if $\{a_j\}$ is any sequence of positive numbers, then

$$\bar{a}_H \le \bar{a}_G \le \bar{a}_A .$$

(hint: use Jensen's inequality)