Statistics 321, Statistical Computing, Home Work 8

Handed Out: Oct. 24, 1996 Due: Oct. 31, 1996

The goal of this is to build understanding of orthogonal function ideas, for a different basis. One such is the "Legendre Polynomials", which form a basis of $L^2(w)$, for $w = 1_{[-1,1]}(x)$ (as with the Fourier basis, there are several possible rescalings that are also commonly found). One way of defining this system, is to start with:

$$P_n(x) = \tfrac{1}{2^n n!}\left(\tfrac{d}{dx}\right)^n \left(x^2-1\right)^n, \; n = 0, 1, 2, \ldots$$

Note that $P_n(x)$ is a polynomial of degree n.

1. Show that the first four of these are given by

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \tfrac{1}{2}(3x^2-1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3-3x).$$

2. Use repeated integration by parts to show that $\{P_n\}_{n=0}^{\infty}$ is an orthogonal system with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

Note: This is why that strange derivative form turns out to be very convenient. Actually most major bases of orthogonal functions have representations in such a derivative form.

Hint: Note that $[\cdots]\left(x^2-1\right)\Big|_{-1}^{1} = 0$ for *any* expression $[\cdots]$.

3. Calculate $\langle P_n, P_n\rangle$, and use it to show that

$$\psi_n(x) = \sqrt{n+\tfrac{1}{2}}\, P_n(x), \; n = 0, 1, 2, \ldots$$

forms an orthonormal system.

Hint: do n integrations by parts, and use the substitution $u = \frac{x+1}{2}$ to get a "beta integral", which can be evaluated by the same methods used to get the normalizing constant for the beta density.

4. The definition given above is very convenient for mathematical manipulations, but it makes actual calculation of the polynomials quite tedious in general. A simpler way to calculate these is through the "recurrence formula":

$$P_n(x) = \tfrac{1}{n}\Big((2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)\Big).$$

This can be proven in general by a fairly tricky induction argument (but is well known, see e.g. part 1 of 8.914 from Gradshteyn and Ryzhik). Simply verify this in the case $n = 2, 3$ for the polynomials given in (1) above. (This type of verification is generally a good idea when using formulas from books, because it provides a check that you are using the formulas correctly).

5. Write a GAUSS program which makes a plot of the basis functions for $n = 0, 1, 2, 3, 4$. To be sure that you have programmed the basis functions correctly:

   a. Check them against the first 4, from part (1) above.

   b. Do a numerical approximation to the integrals which give the inner products. I.e. check that
   $$\langle \psi_n, \psi_m \rangle = 1_{\{n=m\}}.$$

   Only turn in the plot, but be to have a legend, and to choose a careful ordering of the line/symbol types. Think about the similarity of this basis to the Fourier basis.

6. Show that for the function $f_a(x) = e^{ax}$, the coefficients of the orthogonal expansion of f_a are:
   $$\langle f_a, \psi_n \rangle = \sqrt{n + \tfrac{1}{2}} \, \tfrac{(-a)^n}{2^n n!} \Sigma_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (-1)^{n-j} I_j$$
   where $I_j = \int_{-1}^{1} e^{ax} x^{2j} \, dx$ (not quite the gamma function!).

   Hint: Use repeated integrations by parts, and the binomial theorem.

7. Show that the I_j above can be calculated using the recurrence relation
   $$I_j = \tfrac{2}{a} \sinh(a) - \tfrac{4j}{a^2} \cosh(a) + \tfrac{2j(2j-1)}{a^2} I_{j-1},$$
   and the starting value
   $$I_0 = \tfrac{2}{a} \sinh(a).$$

8. Write a GAUSS program which makes plots showing the projections of f_a onto the spans of the sets $\{\psi_n\}_{n=1}^{k}$, for $k = 0, 1, 2, 3, 4$. Include the proportion of the power of f that is explained by each of the projections (in some way). Turn in plots for $a = 1, 4, 10$, and discuss quality of approximation (and the resulting implications for Taylor expansion) in each case.

   Note: A simple and effective way of checking what you are doing (i.e. of getting these fairly complicated formulas right) is: as you write the program, verify your values of the I_n and $\langle f_a, \psi_n \rangle$ against numerical integrals for them.